Week 10

Last time: Let H be a Hilbert space

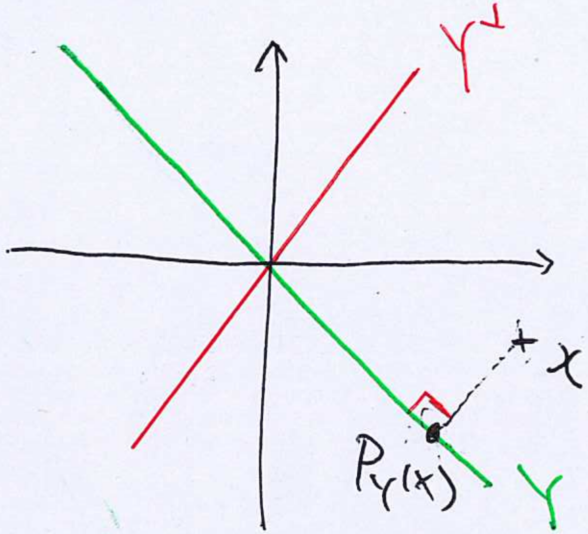
$Y \subseteq H$ be a closed subspace

① If $y_0 \in Y$, $x \in H$, then

$$y_0 = P_Y(x) \iff x - y_0 \in Y^\perp$$

② $H = Y \oplus Y^\perp$

Picture $H = \mathbb{R}^2$



Property of $P_Y : H \to H$ (Y is closed subspace)

① For $x \in H$, $x = y + z$, $y \in Y$, $z \in Y^\perp$

then $P_Y(x) = y$

② P_Y is linear

③ Null space $N(P_Y) = Y^\perp$

Range $R(P_Y) = Y$

④ $P_Y^2 = P_Y$

⑤ $\|P_Y\| = \begin{cases} 1 & Y \neq \{0\} \\ 0 & Y = \{0\} \end{cases}$

Lemma 3.3-6 (Stronger) H Hilbert space, Y is subspace

- $Y^\perp$ is a closed subspace (Inner product is continuous)
- $(Y^\perp)^\perp = \overline{Y}$

Lemma 3.3-7 H Hilbert space, M subset

$$\overline{\operatorname{Span} M} = H \iff M^\perp = \{0\}$$

Pf ($\Rightarrow$) Suppose $\overline{\operatorname{Span} M} = H$

Let $x \in M^\perp \subseteq H = \overline{\operatorname{Span} M}$

$\Rightarrow \exists$ sequence $x_n \in \operatorname{Span} M$ such that

$$x = \lim_{n\to\infty} x_n$$

$$x \in M^\perp \Rightarrow x \in (\operatorname{Span} M)^\perp$$

Why?. If $a_1x_1 + a_2x_2 + \cdots + a_nx_n$

$a_i \in \mathbb{F}$, $x_i \in M$

Then $\langle x, \sum a_i x_i \rangle = \sum \overline{a_i} \langle x, x_i \rangle$

$= \sum \overline{a_i}(0)$

$= 0$

$\Rightarrow \langle x, x_n \rangle = 0$

Inner product is continuous, $\lim_{n\to\infty} x_n = x$

$\Rightarrow \langle x, x \rangle = 0 \quad \Rightarrow x = 0$

$\Rightarrow M^\perp = \{0\}$

($\Leftarrow$) Suppose $M^\perp = \{0\}$

then $(\operatorname{Span} M)^\perp = M^\perp = \{0\}$

$$H = (\operatorname{Span} M)^\perp \oplus (\operatorname{Span} M)^{\perp\perp}$$

$$= \{0\} \oplus \overline{\operatorname{Span} M}$$

$$= \overline{\operatorname{Span} M}$$

Defn H is Hilbert space.

① A subset $M \subseteq H$ is said to be *orthogonal* if $x \perp y$ for $x, y \in M$ and $x \neq y$

② A sequence (x_i) in H is said to be *orthogonal* if $x_i \perp x_j$ for $i \neq j$

③ A subset or a sequence is said to be *orthonormal* if it is orthogonal and every vector in it is a unit vector

i.e. length = 1

Basic Properties

Pythagorean $x_1, x_2, \cdots, x_n \in H$ orthogonal,

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Linear independence

An orthonormal set is linearly independent

Pf Let $x_1, x_2, \cdots, x_n$ be orthonormal

Suppose $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \vec{0}$

Then $\langle \sum_{i=1}^{n} a_i x_i, x_j \rangle = \langle \vec{0}, x_j \rangle = 0$

$$\| $$

$$\sum_{i=1}^{n} a_i \langle x_i, x_j \rangle = a_j$$

Note $\langle x_i, x_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$\Rightarrow$ linearly independent

Projection / Linear combination

Suppose $\{x_1, x_2, \cdots, x_n\} = M$ is orthonormal

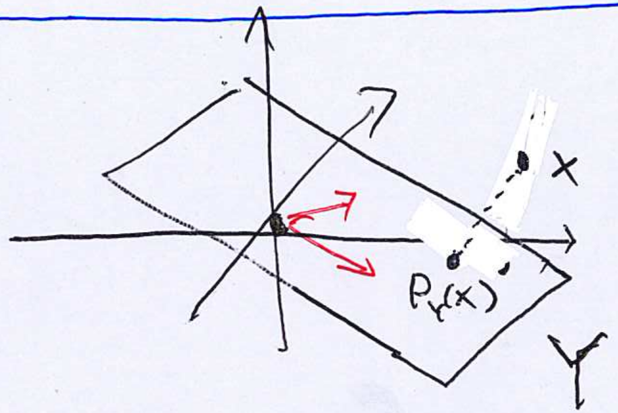
- If $x \in \operatorname{Span} M$, then

$$x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

- In general, if $x \in H$, then

$$P_Y x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

where $Y = \operatorname{span} M$



Pf for $P_Y(x)$

Let $P_Y(x) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

$$x - P_Y(x) \perp Y = \operatorname{Span} M$$

$$\Rightarrow \langle x - P_Y(x), x_i \rangle = 0 \quad \forall i$$

$$\Rightarrow \langle x, x_i \rangle - \sum_{j=1}^{n} \langle a_j x_j, x_i \rangle = 0$$

$$\langle x, x_i \rangle - \sum_{j=1}^{n} a_j \langle x_j, x_i \rangle = 0$$

$$\langle x, x_i \rangle - a_i \langle x_i, x_i \rangle = 0$$

$$\langle x, x_i \rangle = a_i$$

$$\Rightarrow P_Y x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

eg of orthonormal sequence

1. $e_1, e_2, e_3, \cdots, e_n \in \mathbb{R}^n$ or $\mathbb{C}^n$

$$e_i = (0, 0, \cdots 0, \underset{\substack{\uparrow \\ i\text{-th entry}}}{1}, 0, \cdots 0)$$

2. $e_1, e_2, e_3, \cdots \cdots \in \ell^2$

$$e_i = (0, 0, \cdots, 0, \underset{\substack{\uparrow \\ i\text{-th entry}}}{1}, 0, 0, \cdots)$$

3. $L^2[0, 2\pi]$ $\quad \langle f, g \rangle = \int_0^{2\pi} f(t) g(t)\, dt$

$$e_0(t) = \frac{1}{\sqrt{2\pi}}$$

$$e_n(t) = \frac{\cos nt}{\sqrt{\pi}} \quad \text{for } n = 1, 2, 3, \cdots$$

$$\tilde{e}_n(t) = \frac{\sin nt}{\sqrt{\pi}} \quad \text{for } n = 1, 2, 3, \cdots$$

Orthonormal set is easy for computation

eg. Linear combination by inner product

$\boxed{\text{Gram-Schmidt Process}}$

linear indept sequence $\longrightarrow$ Orthonormal sequence

Given linearly indept $w_1, w_2, \cdots, w_n \cdots$

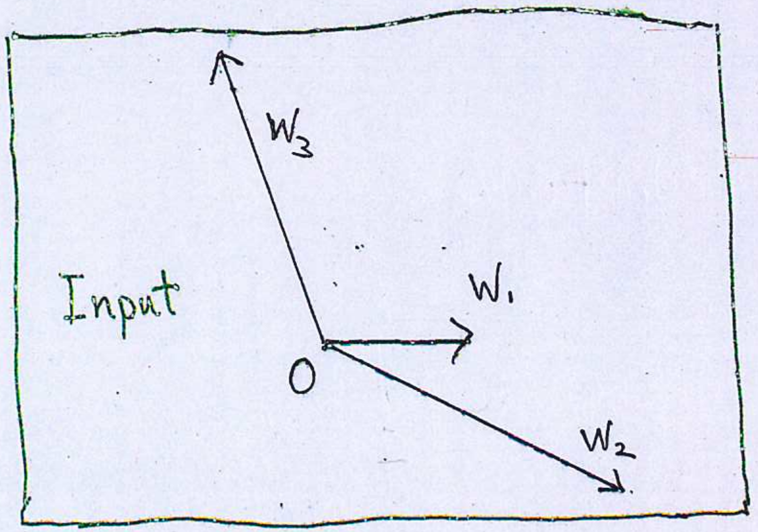
Procedure (Slightly different from book)

1. Let $v_1 = w_1$
2. Let $v_k = w_k - \sum_{j=1}^{k-1} \overbrace{\frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j}^{\mathrm{Proj}_{v_j} w_k}$ for $k \geq 2$
3. Let $e_k = \frac{v_k}{\|v_k\|}$ for $k \geq 1$

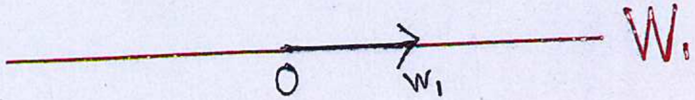
Then $e_1, e_2, e_3 \cdots$ is orthonormal

Picture $n=3$, $\mathbb{F}=\mathbb{R}$

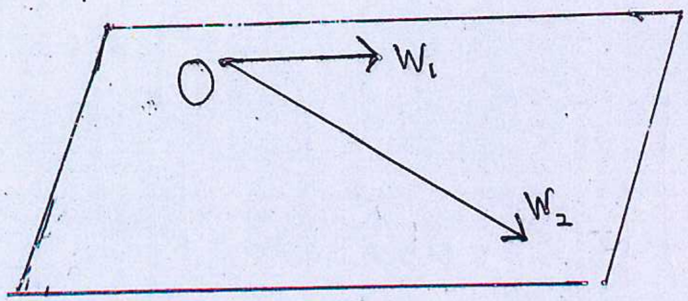
Given lin. indept $S=\{w_1, w_2, w_3\}$



$W_1 = \text{span}\{w_1\}$



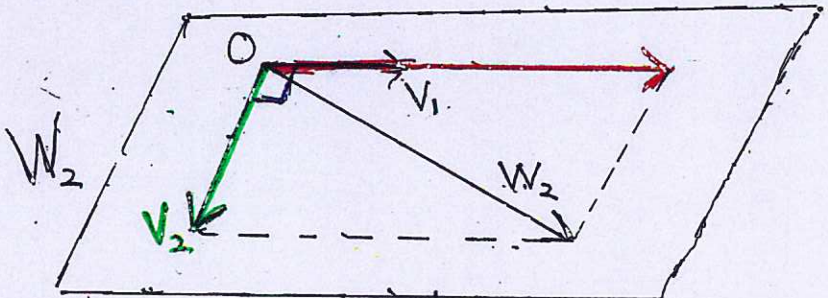
$W_2 = \text{span}\{w_1, w_2\}$



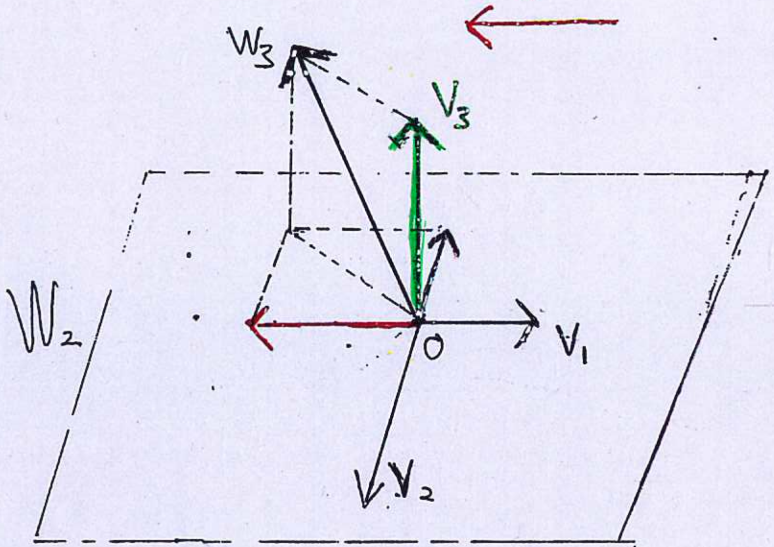
Orthogonalization Process:

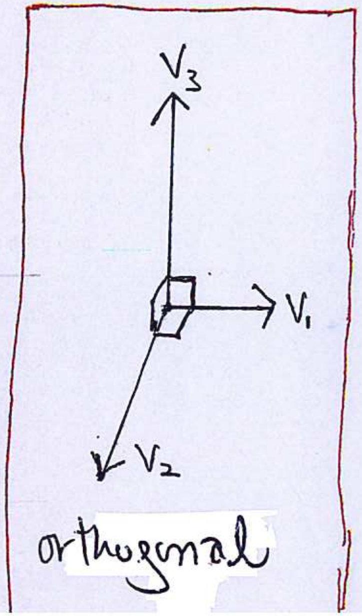
① $v_1 = w_1 = \longrightarrow$

② $$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$$



③ $$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = \uparrow$$



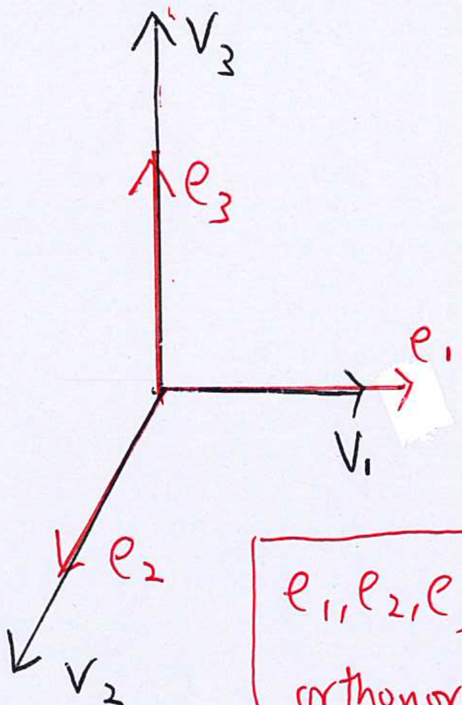


Normalization Process

$$e_1 = \frac{v_1}{\|v_1\|}$$

$$e_2 = \frac{v_2}{\|v_2\|}$$

$$e_3 = \frac{v_3}{\|v_3\|}$$



$\boxed{\begin{array}{c} e_1, e_2, e_3 \\ \text{orthonormal} \end{array}}$

Gram-Schmidt Process

Input : lin indept $w_1, w_2, \ldots$ $\xrightarrow{\text{Orthogonalization}}$ Orthogonal $v_1, v_2, \ldots$ $\xrightarrow{\text{Normalization}}$ Orthonormal $e_1, e_2, \ldots$

eg
$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\2\end{bmatrix} \in \mathbb{R}^3$$

Ans
$$\xrightarrow{\text{Orthogonalization}} \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$\xrightarrow{\text{Normalization}} \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

eg2 $\{1, x, x^2\} \subseteq P_2(\mathbb{R})$ is basis, with $1 = w_1$, $x = w_2$, $x^2 = w_3$

Orthogonalization

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

$$v_1 = w_1 = 1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = w_2 = x$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= w_3 - \boxed{\frac{1}{3}} v_1 - \boxed{0}$$

$$= x^2 - \frac{1}{3}$$

$$\langle w_3, v_2\rangle = \int_{-1}^{1} x^3\,dx = 0$$ odd function

$$\langle w_3, v_1\rangle = \int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}$$

$$\|v_1\|^2 = \langle v_1, v_1\rangle = \int_{-1}^{1} (1)(1)\,dx = 2$$

Normalization

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$e_2 = \frac{v_2}{\|v_2\|} = \frac{v_2}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\, x$$

$$e_3 = \frac{v_3}{\|v_3\|} = \frac{v_3}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

eg3 Consider $y = |x| \in L^2[-1,1]$ $\quad P_2(\mathbb{R}) \subseteq L^2[-1,1]$

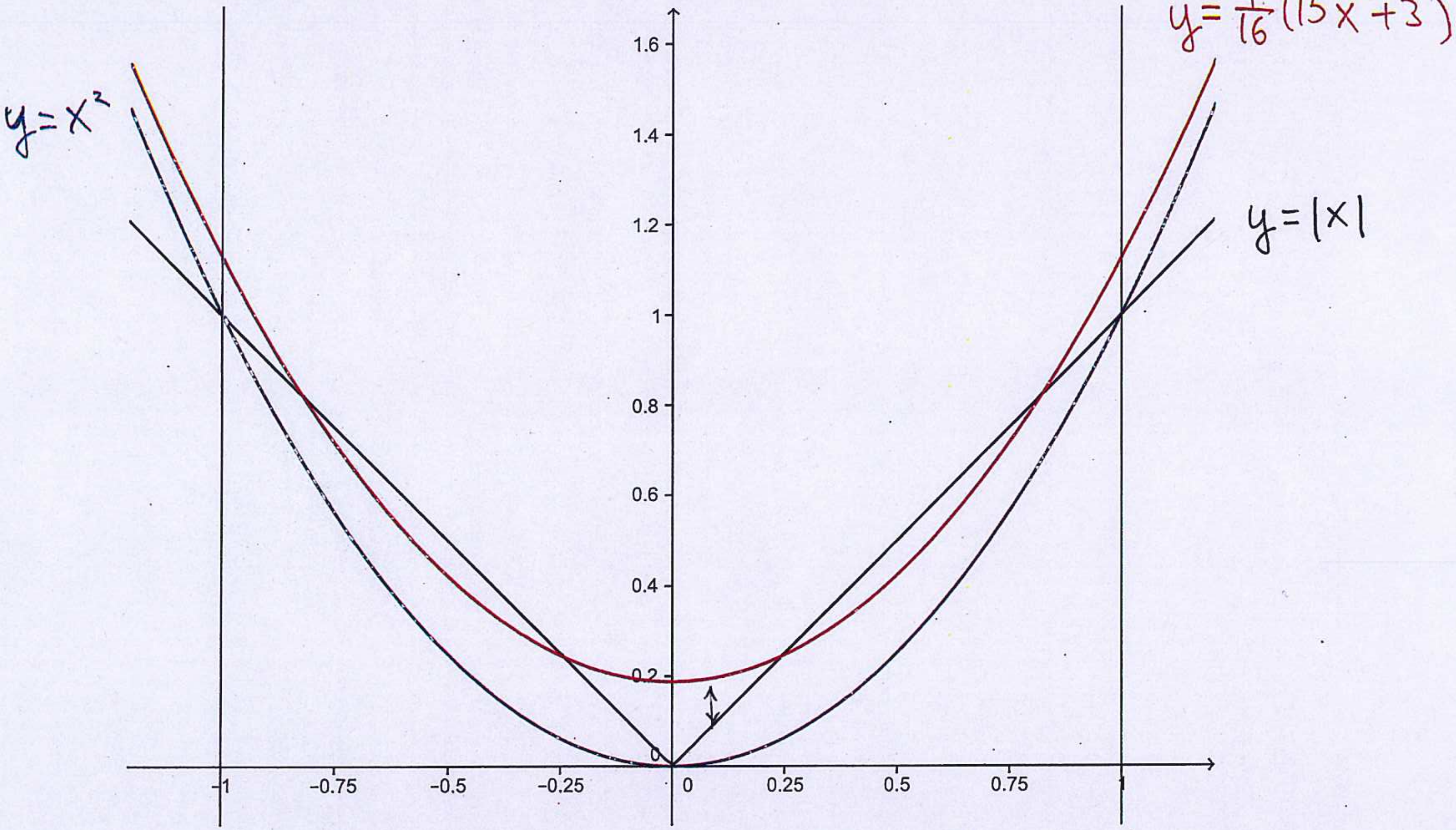
Find orthogonal projection of y onto $P_2(\mathbb{R})$

Sol $\mathrm{Proj}_{P_2(\mathbb{R})}(y)$

$$= \langle y, e_1\rangle e_1 + \langle y, e_2\rangle e_2 + \langle y, e_3\rangle e_3$$

$$= \frac{1}{16}(15x^2 + 3)$$

Approximating $|x|$ by $\deg \leq 2$ polynomials over $[-1,1]$

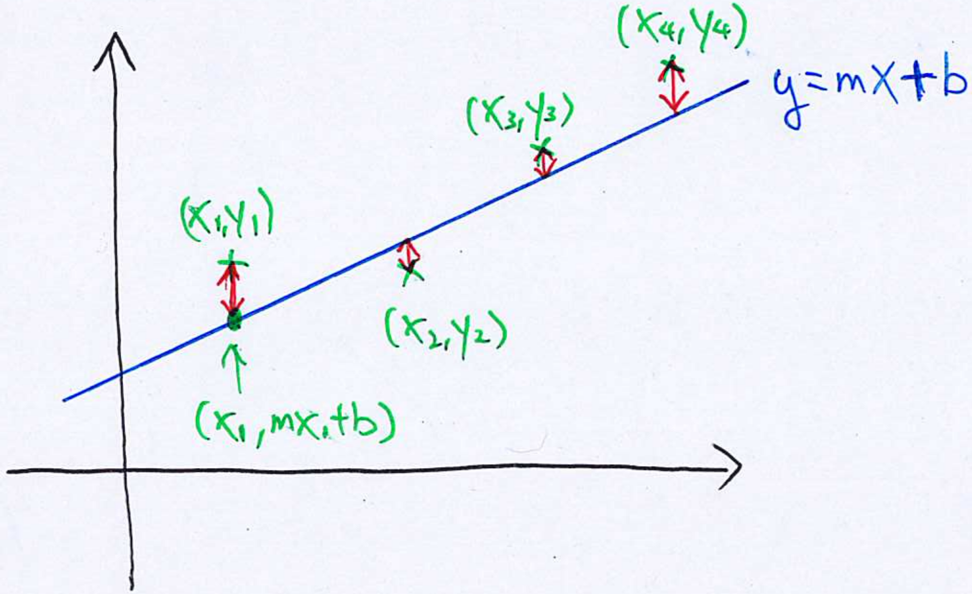


$y = \frac{1}{16}(15x^2+3)$ is our "best" one, found by orthogonal projection of $|x|$ onto $P_2(\mathbb{R})$

$\int_{-1}^{1}\left(|x| - \frac{1}{16}(15x^2+3)\right)^2 dx$ is minimum

Least square Problem

Given $x_1 < x_2 < \cdots < x_n$ and points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \mathbb{R}^2$



Goal: Find a line $y = mx + b$ such that

$$W = \sum_{i=1}^{n} (mx_i + b - y_i)^2$$

is minimum

Ans:

$$m = \frac{(\sum x_k)(\sum y_k) - n \sum x_k y_k}{(\sum x_k)^2 - n \sum x_k^2}$$

$$b = \frac{1}{n}\left(\sum y_k - m \sum x_k\right)$$

Perspective from orthogonal projection

n points $(x_i, y_i) \leftrightarrow q \in P_{n-1}(\mathbb{R})$

$y_i = q(x_i)\ \forall i$

Define an inner product on $P_{n-1}(\mathbb{R})$ by

$$\langle p_1, p_2 \rangle = \sum_{i=1}^{n} p_1(x_i)\, p_2(x_i)$$

Least square problem $\leftrightarrow \operatorname{Proj}_{P_1(\mathbb{R})} q = mx + b$

Remark about Gram-Process on $P(\mathbb{R})$

$\{1, x, x^2, x^3, \cdots\}$ basis for $P(\mathbb{R}) \subseteq L^2[-1,1] = H$

$$\Big\downarrow \text{G.S process}$$

$\{e_0, e_1, e_2, e_3 \cdots\}$

where $$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}(t^2-1)^n \leftarrow \text{degree } n \text{ Legendre polynomial}$$

and $$e_n(t) = \sqrt{\frac{2n+1}{2}}\, P_n(t)$$

Ex Verify $\langle P_n, P_m \rangle = 0$ for $n \neq m$ by integration by parts and reduction formula

$\langle e_n, e_n \rangle = 1$

Thm 3.5-2 Let (e_n) be an orthonormal sequence in a Hilbert space H

① Let $\alpha_i \in \mathbb{F}$, $i = 1, 2, \cdots$: Then

$$\sum_{i=1}^{\infty} \alpha_i e_i \text{ converges (in } H)$$

$$\Longleftrightarrow \sum_{i=1}^{\infty} |\alpha_i|^2 \text{ converges (in } \mathbb{R})$$

$$\Longleftrightarrow (\alpha_i) \in \ell^2$$

② Let $x \in H$, then

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \le \|x\|^2 \quad \text{(Bessel inequality)}$$

and $\displaystyle\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i = \mathrm{Proj}_Y(x)$,

where $Y = \overline{\mathrm{Span}\{e_1, e_2, e_3, \cdots\}}$.

Equality holds $\Longleftrightarrow x \in Y = \overline{\mathrm{Span}\{e_1, e_2, \cdots\}}$

Pf of Bessel inequality

Let $Y_k = \mathrm{Span}\{e_1, e_2, \cdots, e_k\}$

and $y = \mathrm{Proj}_{Y_k}(x)$

$$= \sum_{i=1}^{k} \langle x, e_i \rangle e_i$$

Note that $\|\mathrm{Proj}_{Y_k}\| \le 1$

$\therefore \quad \|y\|^2 \le \|x\|^2$

$$\Rightarrow \sum_{i=1}^{k} |\langle x, e_i \rangle|^2 \le \|x\|^2$$

let $k \to \infty$, $\Rightarrow$ Bessel inequality

Defn Let H be a Hilbert space

An orthonormal set $M \subseteq H$ is said to be total if

$$\overline{\text{Span } M} = H$$

If $\{e_1, e_2, e_3, \cdots\}$ is an orthonormal sequence and is total, then $\forall x \in X$

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$$

by previous theorem

e.g.

$\{1, x, x^2, \cdots\}$ is a basis for $P(\mathbb{R})$

$$\downarrow \text{G.S. Process}$$

$\{e_0, e_1, e_2, \cdots\}$ is an orthonormal sequence and a basis for $P(\mathbb{R})$

Note that every $f \in C[-1,1]$ can be "well-approximated" by a polynomial and $C[-1,1]$ is dense in $L^2[-1,1]$

$\therefore \overline{\text{Span}\{e_0, e_1, e_2, \cdots\}} = \overline{P(\mathbb{R})} = L^2[-1,1]$

$\therefore \{e_0, e_1, e_2, \cdots\}$ is total in $L^2[-1,1]$

$\therefore \forall f \in L^2[-1,1]$

$$f = \sum_{i=0}^{\infty} \langle f, e_i \rangle e_i \quad (\text{in } L^2\text{-norm})$$